N-Body Methods and Practicals

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Basic N-Body Integration

Hermite Code Implementations

Programming Exercises

Introduction to NBODY6

Practicals

NBODY6 Projects

http://www.ast.cam.ac.uk/~ sverre

http://www.sverre.com

Newton's Equations

Force $\mathbf{F}_i = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$

Explicit differentiation

$$\mathbf{F}_i^{(1)} = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} - 3m_j \frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

New solution at $t = \Delta t$

$$\Delta \dot{\mathbf{r}}_i = (\frac{1}{2}\mathbf{F}_i^{(1)}\Delta t + \mathbf{F}_i)\Delta t$$

$$\Delta \mathbf{r}_i = ((\frac{1}{6}\mathbf{F}_i^{(1)}\Delta t + \frac{1}{2}\mathbf{F}_i)\Delta t + \dot{\mathbf{r}}_i)\Delta t$$

Hermite Integration

Taylor series for $\mathbf{F}$ and $\mathbf{F}^{(1)}$

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_0 + \mathbf{F}_0^{(1)}\, t + \frac{1}{2}\mathbf{F}_0^{(2)}\, t^2 + \frac{1}{6}\mathbf{F}_0^{(3)}\, t^3 \\ \mathbf{F}^{(1)} &= \mathbf{F}_0^{(1)} + \mathbf{F}_0^{(2)}\, t + \frac{1}{2}\mathbf{F}_0^{(3)}\, t^2 \end{aligned}$$

Prediction

$$\begin{aligned} \mathbf{r}_j &= ((\frac{1}{6}\mathbf{F}_0^{(1)}\, \delta t_j' + \frac{1}{2}\mathbf{F}_0)\, \delta t_j' + \mathbf{v}_0)\, \delta t_j' + \mathbf{r}_0 \\ \mathbf{v}_j &= ((\frac{1}{2}\mathbf{F}_{\mathbf{0}}^{(1)}\, \delta t_j' + \mathbf{F}_0)\, \delta t_j' + \mathbf{v}_0\, ; \qquad \delta t_j' = t - t_0 \end{aligned}$$

New forces $\qquad \mathbf{F},\ \mathbf{F}^{(1)}$

Higher derivatives

$$\begin{aligned} \mathbf{F}_0^{(3)} &= (2(\mathbf{F}_0 - \mathbf{F}) + (\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\, t)\, \frac{6}{t^3} \\ \mathbf{F}_0^{(2)} &= (-3(\mathbf{F}_0 - \mathbf{F}) - (2\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\, t)\, \frac{2}{t^2} \end{aligned}$$

Corrector for i

$$\begin{aligned} \Delta\mathbf{r}_i &= \frac{1}{24}\mathbf{F}_0^{(2)}\Delta t^4 + \frac{1}{120}\mathbf{F}_0^{(3)}\,\Delta t^5 \\ \Delta\mathbf{v}_i &= \frac{1}{6}\mathbf{F}_0^{(2)}\Delta t^3 + \frac{1}{24}\mathbf{F}_0^{(3)}\,\Delta t^4 \end{aligned}$$

Time-Steps

Basic time-step $\Delta t = \frac{\alpha|\mathbf{r}|}{|\mathbf{v}|}, \quad \Delta t = \frac{\beta|\mathbf{F}|}{|\mathbf{F}^{(1)}|}$

Taylor series $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0^{(1)}\,\Delta t + \frac{1}{2}\mathbf{F}_0^{(2)}\,\Delta t^2 + \ldots$

Natural time-step $\Delta t = \left(\frac{\eta|\mathbf{F}|}{|\mathbf{F}^{(2)}|}\right)^{1/2}, \quad \eta = 0.02$

General expression $\Delta t = \left(\frac{\eta(|\mathbf{F}||\mathbf{F}^{(2)}| + |\mathbf{F}^{(1)}|^2)}{|\mathbf{F}^{(1)}||\mathbf{F}^{(3)}| + |\mathbf{F}^{(2)}|^2}\right)^{1/2}$

Relative criterion Δt independent of mass

Block-steps $\Delta t_n = \frac{\Delta t_1}{2^{n-1}}, \quad \Delta t_1 = 1$

Hierarchical levels $\mathcal{N}_k$ particles with steps Δt_k

Scheduling $i = \min\ (t_j + \Delta t_j)$

Basic Code Structure

Input	Read input parameters
Initial conditions	Generate m, $\mathbf{r}$, $\dot{\mathbf{r}}$
Initialization	$\mathbf{F}$, $\dot{\mathbf{F}}$ & Δt
Scheduling	Block-step distribution
Prediction	All N particles
Force calculation	Forces and derivatives
Particle integration	Sequential $\mathbf{F}\&\dot{\mathbf{F}}$
Corrector	Fourth order
New time-steps	Relative criterion
New block-steps	Determine next group
Results	Cluster parameters

Units

(a) Scaling relations

Given length scale $R_{\rm V}$ in pc and total mass $NM_{\rm S}$ in $M_\odot$

Velocity scaling

$\tilde{V}^* = 1 \times 10^{-5}(GM_\odot/L^*)^{1/2}$ km/s, with $L^* = 3 \times 10^{18}$ cm

Velocity unit $\quad V^* = 6.557 \times 10^{-2}(NM_{\rm S}/R_{\rm V})^{1/2}\,\text{km/s}$

Fiducial time $\quad \tilde{T}^* = (L^{*3}/GM_\odot)^{1/2} = 14.94$ Myr

Time unit $\quad T^* = 14.94\,(R_{\rm V}^3/NM_{\rm S})^{1/2}$ Myr

(b) Conversion from N-body to physical units

$$\tilde{r} = R_{\rm V}\, r \text{ pc}, \quad \tilde{v} = V^*\, v\; km/s, \quad \tilde{t} = T^*\, t \text{ Myr},$$
$$\tilde{m} = NM_{\rm S}\, m\; M_\odot$$

Crossing time $\quad T_{\rm cr} = 2\sqrt{2}\,T^*$ Myr

Scaling of Initial Conditions

Main input	$N,\ N_{\rm b},\ M_{\rm S},\ R_{\rm pc}$
Cluster parameters	optional IMF and Plummer or King model
Initial data	$\tilde{m}_i,\ \tilde{\mathbf{r}}_i,\ \tilde{\mathbf{v}}_i, ...,\ i = 1,\ N$
Total energy	$E = T - U$
Virial theorem	$\mathbf{v}_i = q\,\tilde{\mathbf{v}}_i,\ \ q = \left[\frac{Q_{\rm V} U}{T}\right]^{1/2},\ \ \mathbf{r}_i = \tilde{\mathbf{r}}_i$
Standard units	$G = 1,\ \ \Sigma\, m_i = 1,\ \ E_0 = -0.25$
Standard scaling	$\hat{\mathbf{r}}_i = \frac{\mathbf{r}_i}{S^{1/2}},\ \hat{\mathbf{v}}_i = \mathbf{v}_i\, S^{1/2},\ S = \frac{E_0}{q^2 T - U}$
Astrophysical units	$V^*,\ T^*,\ R^*$ from $M_{\rm tot},\ R_{\rm pc}$
Primordial binaries	split or copy m_i, introduce $a,\ e,\ \Omega$
Force polynomials	$\mathbf{F}_i,\ \dot{\mathbf{F}}_i,\ \Delta t_i, ...,\ i = 1,\ N$
KS regularization	explicit initialization, $R < R_{\rm cl}$

Modification of COMMON

(a) Constant size

Existing dummies	..., $XDUM(10)$, $NDUM(10)$
New variables	$XNEW(2)$, NEW
	..., $XNEW(2)$, $XDUM(8)$, NEW, $NDUM(9)$

(b) Enlargement

Increase COMMON	$COMMON/EXTRA/\ \ A(5)$, B, $NEW(6)$
Add to MYDUMP	$REAL * 4\ \ XNEW$
New COMMON	$COMMON/EXTRA/\ \ XNEW(18)$
Add READ/WRITE	..., $XNEW$