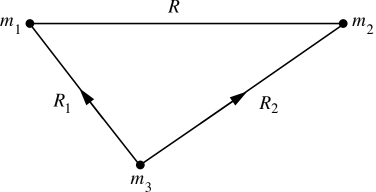

R
m1
m2
R1
R2
m3

Three-Body Regularization

Initial conditions $\mathbf{r}_i,\ \mathbf{p}_i, \quad \mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2)$

Basic Hamiltonian $\mu_{k3} = \dfrac{m_k m_3}{m_k + m_3}$

$$H = \sum_{k=1}^{2} \frac{1}{2\mu_{k3}} \mathbf{p}_k^2 + \frac{1}{m_3} \mathbf{p}_1 \cdot \mathbf{p}_2 - \frac{m_1 m_3}{R_1} - \frac{m_2 m_3}{R_2} - \frac{m_1 m_2}{R}$$

KS transformations $\mathbf{Q}_k^2 = R_k\,, \quad \mathbf{P}_k^2 = 4 R_k \mathbf{p}_k^2, \quad (k = 1, 2)$

Time transformation $dt = R_1 R_2\, d\tau$

Regularized Hamiltonian $\Gamma^* = R_1 R_2\, (H - E_0)$

$$\Gamma^* = \sum_{k=1}^{2} \frac{1}{8\mu_{k3}} R_{3-k}\, \mathbf{P}_k^2 + \frac{1}{16 m_3} \mathbf{P}_1^T \mathbf{A}_1 \cdot \mathbf{A}_2^T \mathbf{P}_2$$
$$- m_1 m_3 R_2 - m_2 m_3 R_1 - \frac{m_1 m_2 R_1 R_2}{|\mathbf{R}_1 - \mathbf{R}_2|} - E_0 R_1 R_2$$

Equations of motion

$$\frac{d\mathbf{Q}_k}{d\tau} = \frac{\partial \Gamma^*}{\partial \mathbf{P}_k}, \qquad \frac{d\mathbf{P}_k}{d\tau} = -\frac{\partial \Gamma^*}{\partial \mathbf{Q}_k}$$

Regular solutions: $R_1 \rightarrow 0$ or $R_2 \rightarrow 0$

Singular term < regular terms: $|\mathbf{R}_1 - \mathbf{R}_2| > \max(R_1, R_2)$

Three-Body Transformations

Coordinates & momenta $\mathbf{q}_k = \tilde{\mathbf{q}}_k - \tilde{\mathbf{q}}_3, \quad \mathbf{p}_k = \tilde{\mathbf{p}}_k$

Regularized coordinates ($q_1 \geq 0$)

$$\begin{aligned} Q_1 &= [\tfrac{1}{2}(|\mathbf{q}_1| + q_1)]^{1/2} \\ Q_2 &= \tfrac{1}{2} q_2 / Q_1 \\ Q_3 &= \tfrac{1}{2} q_3 / Q_1 \\ Q_4 &= 0 \end{aligned}$$

Regularized momenta $\mathbf{P}_k = \mathbf{A}_k \, \mathbf{p}_k$

Basic matrix

$$\mathbf{A}_1 = 2 \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ -Q_2 & Q_1 & Q_4 & -Q_3 \\ -Q_3 & -Q_4 & Q_1 & Q_2 \\ Q_4 & -Q_3 & Q_2 & -Q_1 \end{bmatrix}$$

KS transformations $\mathbf{q}_k = \frac{1}{2} \mathbf{A}_k^{\mathrm{T}} \mathbf{Q}_k$

Physical momenta $\mathbf{p}_k = \frac{1}{4} \mathbf{A}_k^{\mathrm{T}} \mathbf{P}_k / R_k$

Coordinates & momenta

$$\begin{aligned} \tilde{\mathbf{q}}_3 &= -\sum_{k=1}^{2} m_k \mathbf{q}_k / M \\ \tilde{\mathbf{q}}_k &= \tilde{\mathbf{q}}_3 + \mathbf{q}_k \\ \tilde{\mathbf{p}}_k &= \mathbf{p}_k \\ \tilde{\mathbf{p}}_3 &= -(\mathbf{p}_1 + \mathbf{p}_2) \qquad (k = 1, 2) \end{aligned}$$

Post-Newtonian Terms

Equation of motion
$$\frac{d^2\mathbf{r}}{dt^2} = \frac{M}{r^2}\left[(-1+A)\frac{\mathbf{r}}{r} + B\mathbf{v}\right]$$

First-order precession
$$M = m_1 + m_2, \quad \eta = \frac{m_1 m_2}{M^2}$$
$$A_1 = 2(2+\eta)\frac{M}{r} - (1+3\eta)v^2 + \frac{3}{2}\eta\dot{r}^2$$
$$B_1 = 2(2-\eta)\dot{r}$$

Higher-order precession
$$A_2 = ..., \quad B_2 = ..., \quad A_3 = ..., \quad B_3 = ...$$

Gravitational radiation
$$A_{5/2} = \frac{8}{5}\eta\frac{M}{r}\dot{r}\left(\frac{17M}{3r} + 3v^2\right)$$
$$B_{5/2} = -\frac{8}{5}\eta\frac{M}{r}\left(3\frac{M}{r} + v^2\right)$$

Total GR perturbation
$$\mathbf{P}_{GR} = \frac{M}{c^2 r^2}\left[(A_1 + \frac{A_2}{c^2} + \frac{A_{5/2}}{c^3})\frac{\mathbf{r}}{r} + (B_1 + \frac{B_2}{c^2} + \frac{B_{5/2}}{c^3})\mathbf{v}\right]$$

Radiation energy loss
$$\Delta E_{GR} = \frac{m_1 m_2}{M}\int \mathbf{P}_{GR}\cdot\mathbf{v}\,dt$$

GR radiation time-scale
$$t_{\rm GR} = \frac{5}{64}\frac{c^5\, g(e)\, a^4}{X(1+X)\, m_x^3}, \quad c = \frac{3\times 10^5}{V^*}$$

Decision-making graduated PN terms from v/c or $t_{\rm GR}$

PN Elements

Energy $$\epsilon_b = \epsilon_0 + \frac{\epsilon_1}{c^2} + \frac{\epsilon_2}{c^4} + \frac{\epsilon_3}{c^6}, \qquad a = -\frac{M}{2\epsilon_b}$$

$$\epsilon_0 = \frac{1}{2}V^2 - \frac{M}{R}, \qquad \eta = \frac{m_1 m_2}{M^2}$$

$$\epsilon_1 = \frac{1}{2}\frac{M}{R} + \frac{3}{8}(1-3\eta)V^4 + \frac{1}{2}\left((3+\eta)V^2 + \eta\dot{R}^2\right)\frac{M}{R}$$

Lenz vector $$\mathbf{e} = \mathbf{V} \times \mathbf{R} \times \mathbf{V}/M - \mathbf{R}/R$$

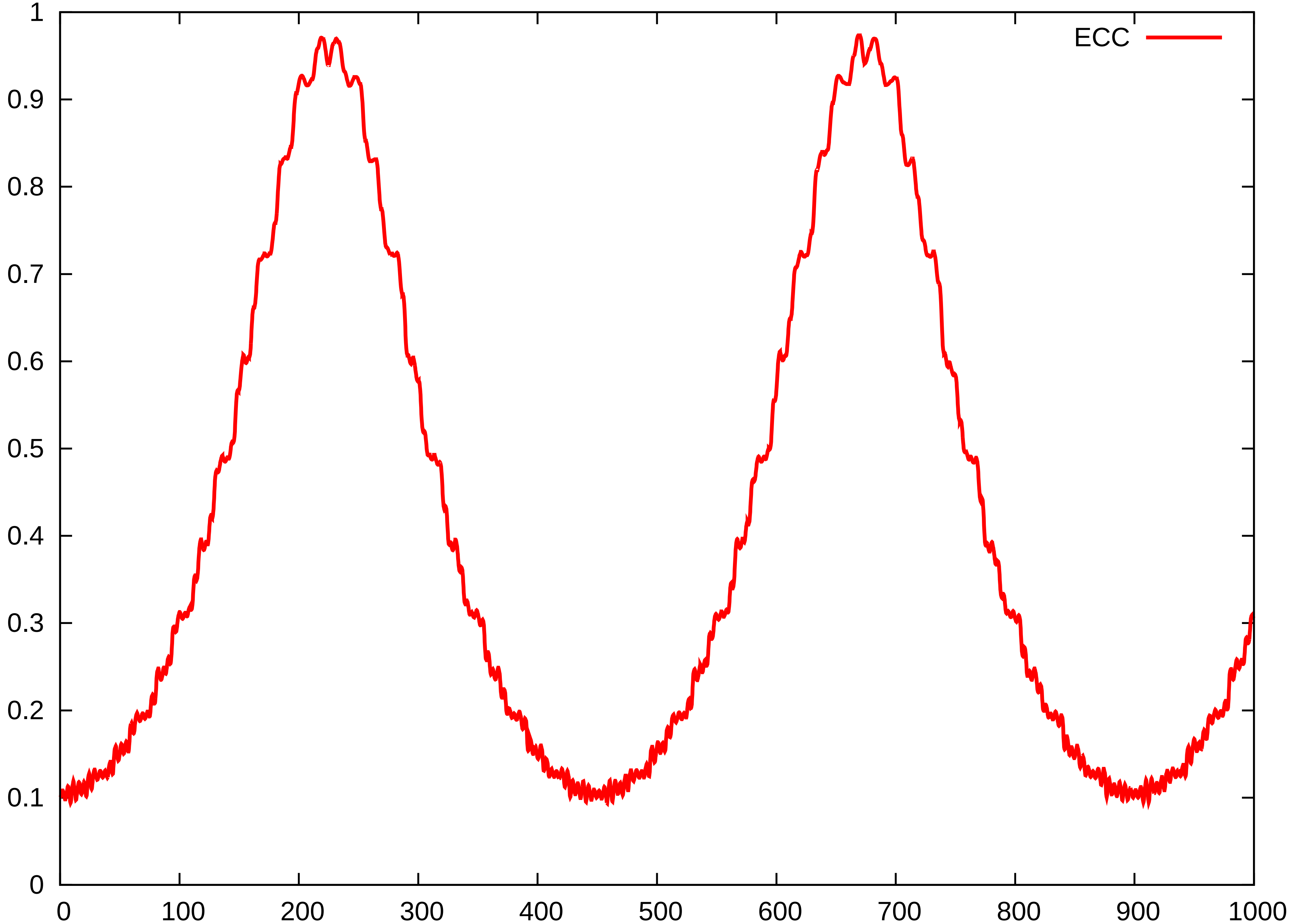
Periapse advance $$\Delta\omega = \frac{6\pi M}{c^2 a(1-e^2)}$$

PN2.5 $$\tau_{GR} = \frac{5g(e)}{64}\frac{a^4 c^5}{X(1+X)m_1^3}, \qquad X = \frac{m_2}{m_1}$$

$$g(e) \simeq \frac{(1-e^2)^{7/2}}{4.5}$$

Angular momentum $$\mathbf{J} = \mathbf{J}_0(1 + f_1/c^2 + f_2/c^4)$$

Eccentricity $$e^2 = (1 - \frac{\mathbf{J}^2}{Ma})$$



ECC
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
100
200
300
400
500
600
700
800
900
1000

Useful Relations

Eccentricity $e^2 = (e\,sin\theta)^2 + (e\,cos\theta)^2$

$$e\,cos\theta = 1 - \frac{R}{a}$$

$$e\,sin\theta = \frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{[(m_k + m_l)a]^{1/2}}$$

Escape $i = 3 - \mathrm{min}_j\{R_j\}, \quad d = |\mathbf{r}_i - \frac{m_3\mathbf{r}_3 + m_j\mathbf{r}_j}{m_3 + m_j}|$

$$R_\mathrm{g} = \frac{\sum m_i m_j}{-E}, \quad \dot{d}^2 > v_\mathrm{cr}^2, \quad d > R_\mathrm{g}, \quad \dot{d} > 0$$

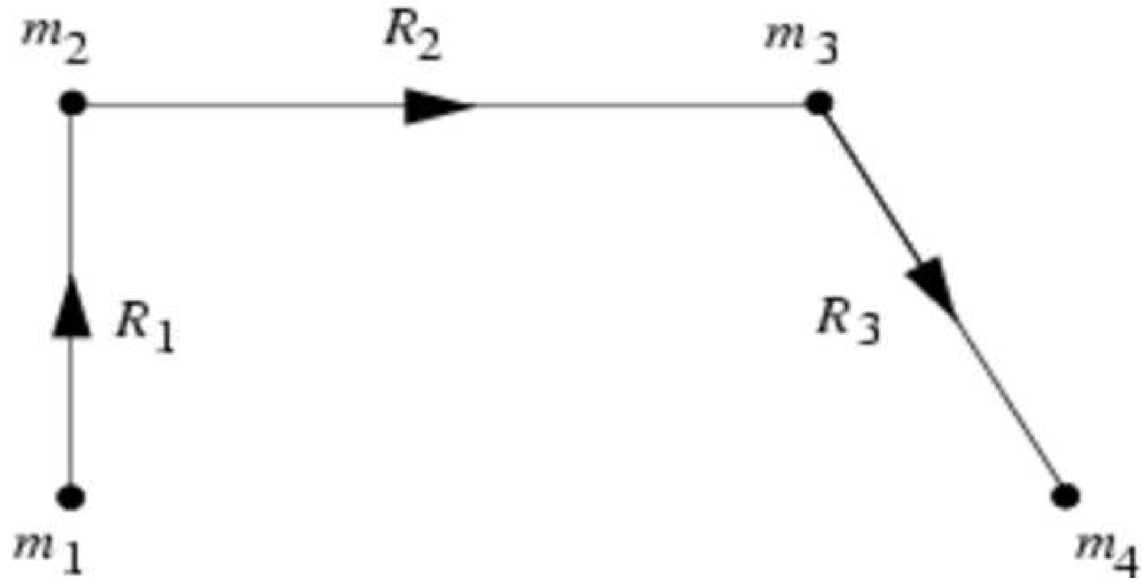
$$v_\mathrm{cr}^2 = 2M\left(\frac{1}{d} + \frac{m_3 m_i Y^2}{d - R_\mathrm{g}}\right), \quad Y = \frac{R_\mathrm{g}}{m_\mathrm{b} d}$$

Kozai cycles $cos^2\psi(1-e^2) = const, \quad q_\mathrm{out} = \frac{m_\mathrm{out}}{m_\mathrm{b}}$

$$T_\mathrm{Kozai} = \frac{T_\mathrm{out}^2}{T_\mathrm{in}}\left(\frac{1 + q_\mathrm{out}}{q_\mathrm{out}}\right)(1-e_\mathrm{out}^2)^{3/2} g(e_\mathrm{in}, \omega_\mathrm{in}, \psi)$$

Lenz vector $\mathbf{e} = \mathbf{V}\times\mathbf{R}\times\mathbf{V}/M - \mathbf{R}/R$

Periapse advance $\Delta\omega = \dfrac{6\pi M}{c^2 a(1 - e^2)}$



m_2
R_2
m_3
R_1
R_3
m_1
m_4

Chain Regularization

Chain vectors $\mathbf{R}_k = \mathbf{r}_{k+1} - \mathbf{r}_k; \quad k = 1, ..., N-1$

Physical momenta $\mathbf{p}_k = m_k \mathbf{v}_k; \quad k = 1, ..., N$

Relative momenta $\mathbf{W}_k = \mathbf{W}_{k-1} - \mathbf{p}_k; \quad k = 2, ..., N-2$

Hamiltonian

$$H = \frac{1}{2} \sum_{k=1}^{N-1} \left(\frac{1}{m_k} + \frac{1}{m_{k+1}} \right) \mathbf{W}_k^2 - \sum_{k=2}^{N-1} \frac{1}{m_k} \mathbf{W}_{k-1} \cdot \mathbf{W}_k - \\ - \sum_{k=1}^{N-1} \frac{m_k m_{k+1}}{R_k} - \sum_{1 \le i \le j-2}^{N} \frac{m_i m_j}{R_{ij}}$$

Equations of motion

$$\frac{d\mathbf{Q}_k}{d\tau} = \frac{\partial \Gamma^*}{\partial \mathbf{P}_k}; \qquad \frac{d\mathbf{P}_k}{d\tau} = - \frac{\partial \Gamma^*}{\partial \mathbf{Q}_k}$$

KS relations $\mathbf{R}_k = \mathcal{L}_k \, \mathbf{Q}_k; \quad \mathbf{W}_k = \mathcal{L}_k \, \mathbf{P}_k / 2\mathbf{Q}_k^2$

Time transformation $dt = g d\tau\,; \quad g = 1/L$

Regularized Hamiltonian $\Gamma^* = g(H - E)$

Regular solutions $R_k \to 0; \qquad k = 1, ..., N-1$