

N -Body Methods and Algorithms

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Basic Integration

Two-Body Regularization

Three-Body Regularization

Post-Newtonian Terms

Practical Aspects

NBODY6 & NBODY7

SSE/GPU Implementations

<http://www.ast.cam.ac.uk/~sverre> <http://sverre.com>

Newton's Equations

$$\text{Force} \quad \mathbf{F}_i = -G \sum_{j=1; j \neq i}^N m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

Explicit differentiation

$$\begin{aligned} \mathbf{F}_i^{(1)} = & -G \sum_{j=1; j \neq i}^N m_j \frac{\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \\ & - 3m_j \frac{(\mathbf{r}_i - \mathbf{r}_j) \cdot (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \end{aligned}$$

New solution at $t = \Delta t$

$$\Delta \dot{\mathbf{r}}_i = \left(\frac{1}{2} \mathbf{F}_i^{(1)} \Delta t + \mathbf{F}_i \right) \Delta t$$

$$\Delta \mathbf{r}_i = \left(\left(\frac{1}{6} \mathbf{F}_i^{(1)} \Delta t + \frac{1}{2} \mathbf{F}_i \right) \Delta t + \dot{\mathbf{r}}_i \right) \Delta t$$

Softening

Potential $\Phi(r) = \frac{Gm}{(r^2 + \epsilon^2)^{1/2}}$

Energy $E = \frac{1}{2}v^2 - \Phi(r) = \text{const}$

Force $\mathbf{F}_i = -G \sum_{j=1; j \neq i}^N m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{(|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2)^{3/2}}$

Force derivative

$$\mathbf{F}_i^{(1)} = -G \sum_{j=1; j \neq i}^N m_j \frac{\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j}{(|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2)^{3/2}} - 3m_j \frac{(\mathbf{r}_i - \mathbf{r}_j) \cdot (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2} \frac{\mathbf{r}_i - \mathbf{r}_j}{(|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2)^{3/2}}$$

Hermite Integration

Taylor series for \mathbf{F} and $\mathbf{F}^{(1)}$

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0^{(1)} t + \frac{1}{2} \mathbf{F}_0^{(2)} t^2 + \frac{1}{6} \mathbf{F}_0^{(3)} t^3$$

$$\mathbf{F}^{(1)} = \mathbf{F}_0^{(1)} + \mathbf{F}_0^{(2)} t + \frac{1}{2} \mathbf{F}_0^{(3)} t^2$$

Prediction

$$\mathbf{r}_j = \left(\left(\frac{1}{6} \mathbf{F}_0^{(1)} \delta t'_j + \frac{1}{2} \mathbf{F}_0 \right) \delta t'_j + \mathbf{v}_0 \right) \delta t'_j + \mathbf{r}_0$$

$$\mathbf{v}_j = \left(\left(\frac{1}{2} \mathbf{F}_0^{(1)} \delta t'_j + \mathbf{F}_0 \right) \delta t'_j + \mathbf{v}_0 \right); \quad \delta t'_j = t - t_0$$

New forces $\mathbf{F}, \mathbf{F}^{(1)}$

Higher derivatives

$$\mathbf{F}_0^{(3)} = (2(\mathbf{F}_0 - \mathbf{F}) + (\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)}) t) \frac{6}{t^3}$$

$$\mathbf{F}_0^{(2)} = (-3(\mathbf{F}_0 - \mathbf{F}) - (2\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)}) t) \frac{2}{t^2}$$

Corrector for i

$$\Delta \mathbf{r}_i = \frac{1}{24} \mathbf{F}_0^{(2)} \Delta t^4 + \frac{1}{120} \mathbf{F}_0^{(3)} \Delta t^5$$

$$\Delta \mathbf{v}_i = \frac{1}{6} \mathbf{F}_0^{(2)} \Delta t^3 + \frac{1}{24} \mathbf{F}_0^{(3)} \Delta t^4$$

Time-Symmetric Method

Reference: Kokubo, Yoshinaga & Makino 1998

Prediction Taylor series to order $\mathbf{F}^{(3)}$

New forces $\mathbf{F}, \mathbf{F}^{(1)}$

Time-symmetric corrector

$$\mathbf{v}_1 = \mathbf{v}_0 + \frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_0)\Delta t - \frac{1}{12}(\dot{\mathbf{a}}_1 - \dot{\mathbf{a}}_0)\Delta t^2$$

$$\mathbf{r}_1 = \mathbf{r}_0 + \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_0)\Delta t - \frac{1}{10}(\mathbf{a}_1 - \mathbf{a}_0)\Delta t^2 + \frac{1}{120}(\dot{\mathbf{a}}_1 + \dot{\mathbf{a}}_0)\Delta t^3$$

Iteration $\mathbf{P}(\text{EC})^n, \quad n = 2, 3$

Block-steps $\Delta t_n = \frac{\Delta t_1}{2^{(n-1)}}, \quad n = 1, \dots, 40$

Reduction Factor 2 if $\Delta t_i < \Delta t_p$

Increase $\Delta t_i > 2\Delta t_p \quad \& \quad \text{dmod}(t, 2 * \Delta t_p) = 0.0$

Time-Steps

Basic time-step $\Delta t = \frac{\alpha|\mathbf{r}|}{|\mathbf{v}|}, \quad \Delta t = \frac{\beta|\mathbf{F}|}{|\mathbf{F}^{(1)}|}$

Taylor series $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0^{(1)} \Delta t + \frac{1}{2}\mathbf{F}_0^{(2)} \Delta t^2 + \dots$

Natural time-step $\Delta t = \left(\frac{\eta|\mathbf{F}|}{|\mathbf{F}^{(2)}|} \right)^{1/2}, \quad \eta = 0.02$

General expression $\Delta t = \left(\frac{\eta(|\mathbf{F}||\mathbf{F}^{(2)}| + |\mathbf{F}^{(1)}|^2)}{|\mathbf{F}^{(1)}||\mathbf{F}^{(3)}| + |\mathbf{F}^{(2)}|^2} \right)^{1/2}$

Relative criterion Δt independent of mass

Block-steps $\Delta t_n = \frac{\Delta t_1}{2^{n-1}}, \quad \Delta t_1 = 1$

Hierarchical levels \mathcal{N}_k particles with steps Δt_k

Scheduling $i = \min (t_j + \Delta t_j)$

Neighbour Scheme

Total force $\mathbf{F}(t) = \sum_{j=1}^n \mathbf{F}_j + \mathbf{F}_d(t)$

Prediction

$$\mathbf{F}(t) = \mathbf{F}_n + \dot{\mathbf{F}}_d(t - t_0) + \mathbf{F}_d(t_0)$$

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_n + \dot{\mathbf{F}}_d$$

Time-scales

$$\Delta t_n \ll \Delta t_d, \quad n \ll N$$

Neighbour sphere $R_s^{\text{new}} = R_s^{\text{old}} \left(\frac{n_p}{n} \right)^{1/3}$

Neighbour selection $|\mathbf{r}_i - \mathbf{r}_j| < R_s$

Derivative corrections $\mathbf{F}_{ij}^{(2)}, \mathbf{F}_{ij}^{(3)}$

Basic Code Structure

Input	Read input parameters
Initial conditions	Generate $m, \mathbf{r}, \dot{\mathbf{r}}$
Initialization	$\mathbf{F}, \mathbf{F}^{(1)}$ & Δt
Scheduling	Form block-step distribution
Prediction	Neighbours or all N
Neighbour integration	Sequential \mathbf{F}_n & $\mathbf{F}_n^{(1)}$
Regular force calc	Sequential total force
New block-step	Determine next group

Direct N-Body Codes

NBODY1	ITS	ϵ	3 – 100
NBODY1H	ITS	ϵ	10 – 10 ⁵
NBODY2	ACS	ϵ	50 – 10 ⁴
NBODY2H	HACS	ϵ	50 – 10 ⁴
NBODY3	ACS	KS & MREG	3 – 100
NBODY5	ACS	KS & MREG	50 – 10 ⁴
NBODY6	HACS	KS & MREG	50 – 10 ⁴
NBODY7	HITS	KS & BH	10 – 10 ⁵
HERMIT	HITS		10 – 10 ⁴
HERMIT2	HITS	& SUN	10 – 10 ⁴
HERMIT4	HITS	KS & SUN	10 – 10 ⁴

MREG: Three-body, four-body & chain regularization

N-Body Codes

<code>nbody0.tar.gz</code>	compact Hermite block-steps
<code>hermit.tar.gz</code>	full Hermite block-steps
<code>nbody1.tar.gz</code>	standard force polynomial
<code>nbody1h.tar.gz</code>	basic Hermite block-steps
<code>nbody2h.tar.gz</code>	AC Hermite block-steps
<code>triple.tar.gz</code>	three-body regularization
<code>chain.tar.gz</code>	chain regularization
<code>nbody6.tar.gz</code>	regularized AC & astrophysics
<code>gpu.tar.gz</code>	GPU software for nbody6