Three-Body Regularization

Initial conditions $\mathbf{r}_i,\ \mathbf{p}_i, \quad \mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2)$

Basic Hamiltonian $\mu_{k3} = \frac{m_k m_3}{m_k + m_3}$

$$H = \sum_{k=1}^{2} \frac{1}{2\mu_{k3}} \mathbf{p}_k^2 + \frac{1}{m_3} \mathbf{p}_1 \cdot \mathbf{p}_2 - \frac{m_1 m_3}{R_1} - \frac{m_2 m_3}{R_2} - \frac{m_1 m_2}{R}$$

KS transformations $\mathbf{Q}_k^2 = R_k\,, \quad \mathbf{P}_k^2 = 4 R_k \mathbf{p}_k^2, \quad (k = 1, 2)$

Time transformation $dt = R_1 R_2\, d\tau$

Regularized Hamiltonian $\Gamma^* = R_1 R_2\, (H - E_0)$

$$\Gamma^* = \sum_{k=1}^{2} \frac{1}{8\mu_{k3}} R_{3-k}\, \mathbf{P}_k^2 + \frac{1}{16 m_3} \mathbf{P}_1^T \mathbf{A}_1 \cdot \mathbf{A}_2^T \mathbf{P_2}$$
$$- m_1 m_3 R_2 - m_2 m_3 R_1 - \frac{m_1 m_2 R_1 R_2}{|\mathbf{R}_1 - \mathbf{R}_2|} - E_0 R_1 R_2$$

Equations of motion

$$\frac{d\mathbf{Q}_k}{d\tau} = \frac{\partial \Gamma^*}{\partial \mathbf{P}_k}, \qquad \frac{d\mathbf{P}_k}{d\tau} = -\frac{\partial \Gamma^*}{\partial \mathbf{Q}_k}$$

Regular solutions: $R_1 \rightarrow 0$ or $R_2 \rightarrow 0$

Singular term < regular terms: $|\mathbf{R}_1 - \mathbf{R}_2| > \max(R_1, R_2)$

Three-Body Transformations

Coordinates & momenta $\mathbf{q}_k = \tilde{\mathbf{q}}_k - \tilde{\mathbf{q}}_3, \quad \mathbf{p}_k = \tilde{\mathbf{p}}_k$

Regularized coordinates ($q_1 \geq 0$)

$$\begin{aligned} Q_1 &= [\tfrac{1}{2}(|\mathbf{q}_1| + q_1)]^{1/2} \\ Q_2 &= \tfrac{1}{2}q_2/Q_1 \\ Q_3 &= \tfrac{1}{2}q_3/Q_1 \\ Q_4 &= 0 \end{aligned}$$

Regularized momenta $\mathbf{P}_k = \mathbf{A}_k \, \mathbf{p}_k$

Basic matrix

$$\mathbf{A}_1 = 2 \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ -Q_2 & Q_1 & Q_4 & -Q_3 \\ -Q_3 & -Q_4 & Q_1 & Q_2 \\ Q_4 & -Q_3 & Q_2 & -Q_1 \end{bmatrix}$$

KS transformations $\mathbf{q}_k = \frac{1}{2}\mathbf{A}_k^{\mathrm{T}}\mathbf{Q}_k$

Physical momenta $\mathbf{p}_k = \frac{1}{4}\mathbf{A}_k^{\mathrm{T}}\mathbf{P}_k/R_k$

Coordinates & momenta

$$\begin{aligned} \tilde{\mathbf{q}}_3 &= -\sum_{k=1}^{2} m_k \mathbf{q}_k / M \\ \tilde{\mathbf{q}}_k &= \tilde{\mathbf{q}}_3 + \mathbf{q}_k \\ \tilde{\mathbf{p}}_k &= \mathbf{p}_k \\ \tilde{\mathbf{p}}_3 &= -(\mathbf{p}_1 + \mathbf{p}_2) \qquad (k = 1, 2) \end{aligned}$$

Perturbed Three-Body Regularization

Regularized Hamiltonian

$$\Gamma^* = R_1 R_2 (H_3 + \mathcal{R} - E)\,, \quad E_3 = E - \mathcal{R}$$

New equations of motion

$$\frac{d\mathbf{Q}_k}{d\tau} = \frac{\partial (R_1 R_2 H_3)}{\partial \mathbf{P}_k}$$

$$\frac{d\mathbf{P}_k}{d\tau} = -(H_3 - E_3)\frac{\partial (R_1 R_2)}{\partial \mathbf{Q}_k} - R_1 R_2 \frac{\partial}{\partial \mathbf{Q}_k}(H_3 + \mathcal{R})$$

External perturbation for Plummer model

$$\frac{\partial \mathcal{R}}{\partial \mathbf{Q}_k} = \sum_{i=1}^{3} \frac{\partial \mathcal{R}}{\partial \mathbf{r}_i} \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_k} \frac{\partial \mathbf{q}_k}{\partial \mathbf{Q}_k}\,, \qquad \frac{\partial \mathcal{R}}{\partial \mathbf{r}_i} = -\frac{m_i M_{\rm p} \mathbf{r}_i}{(r_i^2 + \epsilon^2)^{3/2}}$$

Transformations and c.m. condition $\mathbf{r}_{\rm cm} = \sum m_i \mathbf{r}_i / M$

$$\mathbf{r}_1 = \mathbf{r}_{\rm cm} + (m_2 + m_3)\mathbf{q}_1/M - m_2 \mathbf{q}_2/M$$

$$\mathbf{r}_2 = \mathbf{r}_{\rm cm} - m_1 \mathbf{q}_1/M + (m_1 + m_3)\mathbf{q}_2/M$$

$$\mathbf{r}_3 = \mathbf{r}_{\rm cm} - m_1 \mathbf{q}_1/M - m_2 \mathbf{q}_2/M$$

Application of $\partial \mathbf{r}_i / \partial \mathbf{q}_k$ yields mass ratios

Motion of c.m. $\quad \dfrac{d\mathbf{v}_{\rm cm}}{d\tau} = -R_1 R_2 M_{\rm p} \sum \dfrac{m_i \mathbf{r}_i}{(r_i^2 + \epsilon^2)^{3/2}}$

Energy $\quad \dfrac{dE_3}{d\tau} = -\dfrac{d\mathcal{R}}{d\tau}, \qquad \dfrac{d\mathcal{R}}{d\tau} = \sum_{k=1}^{2} \dfrac{\partial \mathcal{R}}{\partial \mathbf{Q}_k} \dfrac{d\mathbf{Q}_k}{d\tau}$

Basic transformation $\mathbf{q}_k = \mathbf{A}_k^{\mathrm{T}}\mathbf{Q}_k/2$ gives $\partial\mathbf{q}_k/\partial\mathbf{Q}_k = \mathbf{A}_k$

Combining terms

$$\frac{\partial\mathcal{R}}{\partial\mathbf{Q}_k} = -\frac{\mathbf{A}_k m_k}{M}[m_l(\mathbf{F}_k - \mathbf{F}_l) + m_3(\mathbf{F}_k - \mathbf{F}_3)]\,, \quad l = 3 - k$$

Internal energy change

$$\frac{dE_3}{d\tau} = -\frac{d\mathcal{R}}{d\tau}$$

Conversion to known expressions

$$\frac{d\mathcal{R}}{d\tau} = \sum_{k=1}^{2}\frac{\partial\mathcal{R}}{\partial\mathbf{Q}_k}\frac{d\mathbf{Q}_k}{d\tau}$$

Substitution $\quad \dfrac{d\mathbf{Q}_k}{d\tau} = \dfrac{1}{4\mu_{k3}}R_l\mathbf{P}_k + \dfrac{1}{16m_3}\mathbf{A}_k\mathbf{A}_l^{\mathrm{T}}\mathbf{P}_l$

Orthogonality condition

$$\mathbf{A}_k\mathbf{A}_k^{\mathrm{T}} = 4R_k$$

Final energy derivative

$$\frac{d\mathcal{R}}{d\tau} = -\frac{1}{4}\sum_{k=1}^{2}R_l\mathbf{P}_k^{\mathrm{T}}\mathbf{A}_k(\mathbf{F}_k - \mathbf{F}_3)$$

Note $\partial\mathcal{R}/\partial\mathbf{Q}_k$ used for $\mathbf{P}_k'$ and E_3'

Consistency check: $\quad \Delta E = H_3 - E_3$

Post-Newtonian Terms

Equation of motion
$$\frac{d^2\mathbf{r}}{dt^2} = \frac{M}{r^2}\left[(-1+A)\frac{\mathbf{r}}{r} + B\mathbf{v}\right]$$

First-order precession
$$M = m_1 + m_2, \quad \eta = \frac{m_1 m_2}{M^2}$$
$$A_1 = 2(2+\eta)\frac{M}{r} - (1+3\eta)v^2 + \frac{3}{2}\eta\dot{r}^2$$
$$B_1 = 2(2-\eta)\dot{r}$$

Higher-order precession
$$A_2 = ..., \quad B_2 = ..., \quad A_3 = ..., \quad B_3 = ...$$

Gravitational radiation
$$A_{5/2} = \frac{8}{5}\eta\frac{M}{r}\dot{r}\left(\frac{17M}{3r} + 3v^2\right)$$
$$B_{5/2} = -\frac{8}{5}\eta\frac{M}{r}\left(3\frac{M}{r} + v^2\right)$$

Total GR perturbation
$$\mathbf{P}_{GR} = \frac{M}{c^2 r^2}\left[(A_1 + \frac{A_2}{c^2} + \frac{A_{5/2}}{c^3})\frac{\mathbf{r}}{r} + (B_1 + \frac{B_2}{c^2} + \frac{B_{5/2}}{c^3})\mathbf{v}\right]$$

Radiation energy loss
$$\Delta E_{GR} = \frac{m_1 m_2}{M}\int \mathbf{P}_{GR}\cdot\mathbf{v}\,dt$$

PN splitting
$$\mathbf{P}_k = \frac{m_3}{m_k + m_3}\mathbf{P}_{GR}, \quad \mathbf{P}_3 = -\frac{m_k}{m_k + m_3}\mathbf{P}_{GR}$$

Derivatives
$$\mathbf{P}_k' = -t'\frac{\partial\mathcal{R}}{\partial\mathbf{Q}_k}, \quad E_3' = -\sum_{k=1}^{2}\mathbf{Q}_k'\cdot\frac{\partial\mathcal{R}}{\partial\mathbf{Q}_k}$$

PN Decision-Making

Equation of motion $$\frac{d^2\mathbf{r}}{dt^2} = \frac{M}{r^2}\left[(-1 + A)\frac{\mathbf{r}}{r} + B\mathbf{v}\right]$$

Classical form $$\mathbf{F} \;=\; \mathbf{F}_0 \;+\; \frac{\mathbf{F}_2}{c^2} \;+\; \frac{\mathbf{F}_4}{c^4} \;+\; \frac{\mathbf{F}_5}{c^5} \;+\; \frac{\mathbf{F}_6}{c^6}$$

GR radiation time-scale $$t_{\rm GR} \;=\; \frac{5}{64}\frac{c^5\, g(e)\, a^4}{X(1+X)\, m_N^3}, \qquad c = \frac{3\times 10^5}{V^*}$$

$$g(e) \;\simeq\; \frac{(1-e^2)^{7/2}}{4.35}, \qquad X = \frac{m_i}{m_N}$$

Graduated GR effect three stages: $\mathbf{F}_2\,\&\,\mathbf{F}_5,\ \mathbf{F}_4,\ \mathbf{F}_6$

$$t_{\rm GR} \;\leq\; 10\,t,\ \ t,\ \ 0.1\,t$$

Coalescence $$R \;<\; \frac{6\,M}{c^2}$$

Energy check $$E_{\rm tot} \;-\; \int \mathbf{P}_{\rm GR}\cdot\mathbf{v}\,dt \;=\; {\rm const}$$

PN Elements

Energy $$\epsilon_b = \epsilon_0 + \frac{\epsilon_1}{c^2} + \frac{\epsilon_2}{c^4} + \frac{\epsilon_3}{c^6}, \qquad a = -\frac{M}{2\epsilon_b}$$

$$\epsilon_0 = \frac{1}{2}V^2 - \frac{M}{R}, \qquad \eta = \frac{m_1 m_2}{M^2}$$

$$\epsilon_1 = \frac{1}{2}\frac{M}{R} + \frac{3}{8}(1-3\eta)V^4 + \frac{1}{2}\left((3+\eta)V^2 + \eta\dot{R}^2\right)\frac{M}{R}$$

Lenz vector $$\mathbf{e} = \mathbf{V} \times \mathbf{R} \times \mathbf{V}/M - \mathbf{R}/R$$

Periapse advance $$\Delta\omega = \frac{6\pi M}{c^2 a(1-e^2)}$$

PN2.5 $$\tau_{GR} = \frac{5g(e)}{64}\frac{a^4 c^5}{X(1+X)m_1^3}, \qquad X = \frac{m_2}{m_1}$$

$$g(e) \simeq \frac{(1-e^2)^{7/2}}{4.5}$$

Angular momentum $$\mathbf{J} = \mathbf{J}_0(1 + f_1/c^2 + f_2/c^4)$$

Eccentricity $$e^2 = (1 - \frac{\mathbf{J}^2}{Ma})$$

PN Project

Initial conditions	Binary + distant body
Elements	High and low eccentricity
PN order	Coalescence times
Case study	Mercury's perihelion motion
Orbit	a = 0.3871, e = 0.205
Periapse advance	$\Delta\omega = \dfrac{6\pi M}{c^2 a(1-e^2)}$

TRIPLE2 Features

Time reversal	Strict accuracy test: $\sigma_{\rm x}, \ \sigma_{\rm v}$
`X11` movie	`make xtriple`
`PGPLOT` movie	`make ptriple`
External perturbation	Plummer model: $M_{\rm p}, \ a_{\rm p}$
Closest encounter	Osculating two-body separation
Physical collision	Iteration for small R_k (project)
Physical units	Introduce $M^*, \ L^*, \quad \Rightarrow T^*, \ V^*$
Post-Newtonian terms	$c = \dfrac{3 \times 10^5}{V^*}$