N-Body Methods and Algorithms

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Basic Integration

Two-Body Treatments

Three-Body Regularization

Post-Newtonian Terms

Wheel-Spoke Regularization

Chain Regularization

SSE/GPU Implementation

Practical Aspects

NBODY6

Newton's Equations

Force $\mathbf{F}_i = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$

Explicit differentiation

$$\mathbf{F}_i^{(1)} = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} - 3 m_j \frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

New solution at $t = \Delta t$

$$\Delta \dot{\mathbf{r}}_i = (\frac{1}{2}\mathbf{F}_i^{(1)}\Delta t + \mathbf{F}_i)\Delta t$$

$$\Delta \mathbf{r}_i = ((\frac{1}{6}\mathbf{F}_i^{(1)}\Delta t + \frac{1}{2}\mathbf{F}_i)\Delta t + \dot{\mathbf{r}}_i)\Delta t$$

Softening

Potential $\quad \Phi(r) = \frac{Gm}{(r^2 + \epsilon^2)^{1/2}}$

Energy $\quad E = \frac{1}{2}v^2 - \Phi(r) = \text{const}$

Force $\quad \mathbf{F}_i = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{(|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2)^{3/2}}$

Force derivative

$$\mathbf{F}_i^{(1)} = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j}{(|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2)^{3/2}}$$
$$- 3m_j \frac{(\mathbf{r}_i - \mathbf{r}_j) \cdot (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2} \frac{\mathbf{r}_i - \mathbf{r}_j}{(|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2)^{3/2}}$$

Hermite Integration

Taylor series for $\mathbf{F}$ and $\mathbf{F}^{(1)}$

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_0 + \mathbf{F}_0^{(1)}\,t + \frac{1}{2}\mathbf{F}_0^{(2)}\,t^2 + \frac{1}{6}\mathbf{F}_0^{(3)}\,t^3 \\ \mathbf{F}^{(1)} &= \mathbf{F}_0^{(1)} + \mathbf{F}_0^{(2)}\,t + \frac{1}{2}\mathbf{F}_0^{(3)}\,t^2\end{aligned}$$

Prediction

$$\begin{aligned}\mathbf{r}_j &= ((\frac{1}{6}\mathbf{F}_0^{(1)}\,\delta t'_j + \frac{1}{2}\mathbf{F}_0)\,\delta t'_j + \mathbf{v}_0)\,\delta t'_j + \mathbf{r}_0 \\ \mathbf{v}_j &= ((\frac{1}{2}\mathbf{F}_{\mathbf{0}}^{(\mathbf{1})}\,\delta t'_j + \mathbf{F}_0)\,\delta t'_j + \mathbf{v}_0\,; \qquad \delta t'_j = t - t_0\end{aligned}$$

New forces $\qquad \mathbf{F},\ \mathbf{F}^{(1)}$

Higher derivatives

$$\begin{aligned}\mathbf{F}_0^{(3)} &= (2(\mathbf{F}_0 - \mathbf{F}) + (\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\,t)\,\frac{6}{t^3} \\ \mathbf{F}_0^{(2)} &= (-3(\mathbf{F}_0 - \mathbf{F}) - (2\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\,t)\,\frac{2}{t^2}\end{aligned}$$

Corrector for i

$$\begin{aligned}\Delta\mathbf{r}_i &= \frac{1}{24}\mathbf{F}_0^{(2)}\Delta t^4 + \frac{1}{120}\mathbf{F}_0^{(3)}\,\Delta t^5 \\ \Delta\mathbf{v}_i &= \frac{1}{6}\mathbf{F}_0^{(2)}\Delta t^3 + \frac{1}{24}\mathbf{F}_0^{(3)}\,\Delta t^4\end{aligned}$$

Time-Steps

Basic time-step $$\Delta t = \frac{\alpha|\mathbf{r}|}{|\mathbf{v}|}, \quad \Delta t = \frac{\beta|\mathbf{F}|}{|\mathbf{F}^{(1)}|}$$

Taylor series $$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0^{(1)}\Delta t + \frac{1}{2}\mathbf{F}_0^{(2)}\Delta t^2 + \ldots$$

Natural time-step $$\Delta t = \left(\frac{\eta|\mathbf{F}|}{|\mathbf{F}^{(2)}|}\right)^{1/2}, \quad \eta = 0.02$$

General expression $$\Delta t = \left(\frac{\eta(|\mathbf{F}||\mathbf{F}^{(2)}| + |\mathbf{F}^{(1)}|^2)}{|\mathbf{F}^{(1)}||\mathbf{F}^{(3)}| + |\mathbf{F}^{(2)}|^2}\right)^{1/2}$$

Relative criterion Δt independent of mass

Block-steps $$\Delta t_n = \frac{\Delta t_1}{2^{n-1}}, \quad \Delta t_1 = 1$$

Hierarchical levels $\mathcal{N}_k$ particles with steps Δt_k

Scheduling $i = \min\,(t_j + \Delta t_j)$

Neighbour Scheme

Total force $\mathbf{F}(t) = \sum_{j=1}^{n} \mathbf{F}_j + \mathbf{F}_d(t)$

Prediction

$$\mathbf{F}(t) = \mathbf{F}_n + \dot{\mathbf{F}}_d(t - t_0) + \mathbf{F}_d(t_0)$$

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_n + \dot{\mathbf{F}}_d$$

Time-scales

$$\Delta t_n << \Delta t_d, \qquad n << N$$

Neighbour sphere $R_s^{\text{new}} = R_s^{\text{old}} \left(\frac{n_p}{n}\right)^{1/3}$

Neighbour selection $|\mathbf{r}_i - \mathbf{r}_j| < R_s$

Derivative corrections $\mathbf{F}_{ij}^{(2)}, \mathbf{F}_{ij}^{(3)}$

Basic Code Structure

Input	Read input parameters
Initial conditions	Generate m, $\mathbf{r}$, $\dot{\mathbf{r}}$
Initialization	$\mathbf{F}$, $\mathbf{F}^{(1)}$ & Δt
Scheduling	Form block-step distribution
Prediction	Neighbours or all N
Neighbour integration	Sequential $\mathbf{F}_n \& \mathbf{F}_n^{(1)}$
Regular force calc	Sequential total force
New block-step	Determine next group

N-Body Codes

nbody0.tar.gz	compact Hermite block-steps
hermit.tar.gz	full Hermite block-steps
nbody1.tar.gz	standard force polynomial
nbody1h.tar.gz	basic Hermite block-steps
nbody2h.tar.gz	AC Hermite block-steps
triple.tar.gz	three-body regularization
chain.tar.gz	chain regularization
nbody6.tar.gz	regularized AC & astrophysics
gpu.tar.gz	GPU software for nbody6

N-Body Websites

http://www.ast.cam.ac.uk/~sverre

http://www.sverre.com

http://www.cambody.org

email: sverre@ast.cam.ac.uk